Tentamen Functionaalanalyse
25/11/04

1. (a) Let $F : L^2[0,\pi] \to \mathbb{C}$ be defined by

$$F(f) := \int_0^\pi (\sin(t) + \cos(t))\, f(t)dt, \qquad f \in L^2[0,\pi].$$

Show that F is linear, and determine $\|F\|$.

(b) Let $G : L^2[0,\pi] \to \mathbb{C}$ be a bounded linear functional defined on $L^2[0,\pi]$. Does there exist some $g \in L^2[0,\pi]$ such that G is of the form

$$G(f) = 2\pi i \int_0^\pi e^{it} f(t) g(t) dt, \quad f \in L^2[0,\pi]?$$

Justify the answer.

2. Let T be a bounded linear operator from a Banach space $\mathfrak{B}$ into itself with domain $\operatorname{dom} T \subset \mathfrak{B}$. Show that T is closed if and only if $\operatorname{dom} T$ is closed.

3. Let $\mathfrak{H}$ be a Hilbert space and let T and S be linear operators on $\mathfrak{H}$ for which

$$(Tf, g) = (f, Sg), \quad f, g \in \mathfrak{H}.$$

Show that T, S are bounded operators, and that $S = T^*$.
Hint. Use the closed graph theorem.

4. Let E be an infinite-dimensional normed space. Let $x \in E$ with $\|x\| = 1$, and let $U = \operatorname{span}\{x\}$. Let $\ell : U \to \mathbb{C}$ be the linear functional on U such that $\ell(x) = i + 2$.

Does there exist some $L \in E'$ (E' is the dual space of E) such that ℓ is the restriction of L on U: $L|_U = \ell$, and

(a) $\|L\| = 2$?
(b) $\|L\| = \sqrt{5}$?
(c) $\|L\| \geq 5$?

Justify the answers!